MATHEMATICAL UNDERSTANDING IN TRANSITION FROM KINDERGARTEN TO PRIMARY SCHOOL: PLAY AS BRIDGE BETWEEN TWO EDUCATIONAL INSTITUTIONS

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German kindergarten and primary school are characterized by different conditions concerning the organization of learning processes. This places particular demands on the arrangement of linked and coherent mathematical learning environments in transition. Particularly the relational understanding of numbers is one important objective for mathematical learning during the transition. In our qualitative study we developed three complementary playing and learning environments and observed with 20 children how they explore and discuss relationships between numbers in the last year of kindergarten and the first year of school. We focus on the institutional similarities and differences and discuss them with regard to the arrangement of playing and learning environments for kindergarten and primary school.

LEARNING MATHEMATICS BEFORE AND AT THE BEGINNING OF FORMAL SCHOOLING

It is established that mathematical learning processes of children start long before formal schooling in an informal way and take place in other learning locations, e.g. kindergarten. Thus, the arrangement of linked and coherent education processes gains importance and questions arise as to how to cope with the changing learning situations form kindergarten to primary school: In kindergarten, mathematical learning situations arise in play and daily life in the context of both free and guided activities. In primary school, however, learning situations are more formally arranged in the context of substantial learning environments and geared to standards. This demands coping with these discontinuities in a productive way as well as ensuring continuities concerning the learning processes (cf. Roßbach, 2010). In the last years, cooperation activities between teachers and educators from kindergarten and primary school have occurred, with a special focus on becoming familiar with the new learning location. Cooperation activities concerning the coordination and arrangement of academic learning processes are still rare, although they seem to have a positive impact on children’s learning (cf. Ahtola et al., 2011). From the mathematics didactics perspective, the coherence of learning arrangements with respect to content, objectives and requirements is important (cf. e.g. Clements, 2004). While it is discussed how to build on the individual learning experience in primary school, recommendations for the arrangement of mathematical learning situations in kindergarten are developed (cf. e.g. Clements, Sarama & DiBase, 2004; Gasteiger, 2012).

So far, explicit development studies are rare concerning the design of mathematical learning environments which focus on mathematical relations in kindergarten and primary school in due consideration of the particular institutional conditions. This
paper takes up on this research desideratum and focuses both on how to design linked complementary learning environments for kindergarten and primary school and how children explore, identify and discuss mathematical relations within these learning environments.

**Mathematics Learning in Kindergarten and Primary School**

With regard to the development of mathematical understanding, the acquisition of the concept of numbers is of central importance: Early learning processes concerning quantity-number competencies seem to have a lasting effect on mathematical learning at school (cf. Krajewski & Schneider, 2009). Krajewski and Schneider (2009, p. 515) characterize the acquisition of early quantity-number competencies over three levels: According to their model, number-word sequence and an understanding of quantity develop isolated from each other (Level I). Only by linking this »Basic numerical skills« number words gain quantitative meaning (Level II). On level II, children become aware that number-words can be used to describe discrete quantities and that quantities can be determined by counting (»Quantity-number concept«). An understanding of »Number relationships« develops when children realize that numbers relate to other numbers and that numbers can be used to describe these relations (Level III). Two main relations are distinguished:

1. Relation of composition and decomposition: numbers can be described by compositions of other numbers (parts) \(5=3+2\) or can be decomposed into numbers.

2. Relation of difference: numbers can describe differences between two numbers (the difference between 3 and 5 is 2, \(5-3=2\)).

From the developmental psychology point of view, the relational understanding has a meaningful role for children’s further mathematical learning (cf. Langhorst, Ehlert & Fritz, 2012; Krajewski & Schneider, 2009; Resnick, 1983). From the mathematics didactics perspective, the ability to identify and use relations between single numbers by counting and calculating is highlighted as a central objective for learning processes in kindergarten, which should be continued in primary school (cf. Wittmann & Müller, 2009).

However, also from the epistemological point of view the relational understanding is important: Mathematical concepts are not concrete, but characterized by relationships between concrete and abstract objects (cf. Steinbring, 2005; Nührenbörger & Steinbring, 2009). Consequently, mathematical concepts acquire meaning if children deal with the concrete and later abstract objects in an active way and construe relationships between the objects. In transition from kindergarten to primary school, the connection of experiences with concrete objects and systematic abstract examinations is challenging (e.g. Hasemann, 2005). This connection succeeds if children overcome the concrete specific situation: »They are requested to see, interpret or discover ›something else‹, another structure, in the situation« (Steinbring, 2005, p. 82). Accordingly, children have to manage both: They have to identify general structures in
the specific situation and at the same moment consider the specific nature of the situation.

For construing, identifying and using relations between numbers, interaction and negotiation processes are relevant. When children are encouraged to express and discuss their ways of thinking and acting, they get the chance to reason about mathematical meaning and construct mathematical knowledge (cf. e.g. Nührenbörger & Steinbring, 2009).

Playing and Mathematics Learning

Playing respectively play-based learning is one basic approach in kindergarten. Essentially, playing can be seen as social interactive activity, which is characterized by certain rules and roles and can be done repeatedly (Oerter, 1999).

Several authors have stressed the importance of play and playfulness for early mathematics learning. Ginsburg (2006) observed children in kindergarten and categorized different types of mathematical play: Two of these types are »Mathematics Embedded in Play« and »Play Centering on Mathematics«. Play is centering on mathematics if the objects of play are mathematical pattern and structures. Playing, thus, is characterized by dealing with mathematical relations. In contrary, mathematics embedded in play can arise by playing e.g. mathematical-rich games. Playing the game is the focused activity and mathematical activities happen casually while playing.

Mathematics embedded in play in kindergarten: Some studies evidence that children achieve basic mathematical competencies by playing games in kindergarten (e.g. Ramani & Siegler, 2008; Stebler, Vogt, Wolf, Hauser & Rechsteiner, 2013). Stebler et al. (2013) demonstrate, exemplified by the game »Shut the box«, that games can provide a meaningful context for mathematical activity and support individual mathematical strategies. Schuler (2011) emphasizes the importance of the »conversational management« of educators for the development of the mathematical potential in play situations. Supporting guiding activities of adults (»guided play«, Hirsh-Pasek, Golinkoff, Berk & Singer, 2009) involve not only the organization of playing activities – e.g. selecting appropriate materials and games – but also stimulating discussions and asking for mathematical reflection and reasoning, oriented on children’s ability and on the playing process (c.f. Ginsburg, 2006; Pramling Samuelson & Asplund Carlsson, 2008; van Oers, 2010). Van Oers (2010) points out how the role-play of children can be used as meaningful context for further mathematical learning. With the help of game recordings, for example, more systematic learning processes can be encouraged.

Mathematics learning in primary school: In primary school, mathematical learning processes are prepared and structured systematically by means of challenging tasks or »substantial learning environments« (Wittmann, 2001). Substantial learning environments, for example, represent »central objectives, contents and principles of teaching mathematics« (ibid, 2) and promote rich mathematical activities. Thus, con-
tent equivalent and integrated learning arrangements, which can be easily adapted to their individual prerequisites, can be provided for all children.

**Play centering on mathematics in primary school:** But also formal learning in school can be playful: »Play Centering on Mathematics« (Ginsburg, 2006) occurs in operative discovering, exploring and inventing patterns and structures. Children play with mathematical objects and explore mathematical relations: They change objects and discover the impact of change and how to react to these changes (cf. Steinweg, 2001). To develop »Play Centering on Mathematics«, an understanding of mathematics is important, as described by Perry & Dockett (2010, p. 717): »If mathematics is as much about understanding connections, processes and possibilities as it is about knowing facts, then play and mathematics have much in common.« Playful mathematical identification and construction of relations between numbers provide a basis for reflecting and discussing relations amongst children and guiding adults.

**Complementary playing and learning environments:** For the design of mathematical learning environments in transition, the different institutional conditions of kindergarten and primary school have to be considered: Similar to the substantial learning environment (SLE) in primary school, a substantial playing environment (SPE) in kindergarten provides rich mathematical experiences and activities by playing games (cf. Stebler et al., 2013). SLE and SPE are complementary if basic sustainable mathematical contents and materials are addressed in kindergarten and are therefore picked up and continued in primary school.

While current studies focus on the importance of playing for mathematics learning in kindergarten, there are fewer findings on how to use mathematical play and playing as a bridge between kindergarten and primary school and how mathematical learning processes are performed in kindergarten and primary school. In the following, children’s interactive learning processes are studied with regard to these questions:

(I) How do children discover elementary numerical relationships in the transition from kindergarten to primary school in the interactive context of the learning situations?

(II) Which similarities and differences can be pointed out concerning the mathematical construction process in kindergarten and primary school?

**METHODS**

As part of the study, three arithmetic based complementary learning environments (CLE) for kindergarten and primary school were designed which focus on the exploration of relations (cf. Nührenbörger & Tubach, 2012). In spring and summer, children in the last year of kindergarten were involved in these CLEs in the context of playing environments. At the beginning of the first school year (in autumn and winter), they got involved again, but now in the context of learning environments. In total, about 20 children were observed by video over two survey cycles dealing with the playing and learning environments in kindergarten and primary school. In each cycle, about four educational employees of different kindergartens and two teachers of primary school were introduced to the particular playing and learning environments be-
fore. The video observation of authentic situations is complemented by qualitative interviews of two children after the playing and learning situation in kindergarten and in the classroom.

The survey method is oriented at Cobb’s et al. method of »design experiments« (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003). This includes the (usually iterative) exemplary testing of teaching and learning arrangements or SLEs with learners e.g. classroom experiments. The purpose of the experimental testing and analyzing of teaching and learning arrangements is to investigate which learning processes are initiated and by what they are supported. This gives answers to how the arrangement has to be optimized in order to support further or deeper learning processes. In a cycling process, a prospective and reflective perspective is adopted on the learning process: »Prototypically, design experiments entail both ›engineering‹ particular forms of learning and systematically studying those forms of learning within the context defined by the means of supporting them« (Cobb et al., 2003, p. 9).

According to this approach, in the present research, the above-mentioned three CLEs are designed to encourage children to reason about mathematical relations as well as to investigate children’s learning processes within these learning environments. The ultimate goal of these design experiments is the development of interactive, promoting and linked learning environments for kindergarten and primary school, as well as the development of local theories concerning the interactive process of understanding and establishing intersubjectivity between children.

**Construction of the learning environment »Who has more?«**

In the following, one of three designed playing and learning environments is exemplified: »Who has more?« is a game for two children. Each child gets a wooden block of five, a ten-frame and a die (with the numbers 0 to 5) in the color blue or red. The game material is completed by small round gaming pieces, called counters (with a red and an alternate blue side). The structured materials up to 5 and 10 (ten-frame and block of five) enable the interpretation of numbers in their relationship to 5 and 10 (cf. Flexer, 1986). Number 0 is represented as amount of pips on the die as well as difference of two equal numbers.

**Rule of the game:** Both players roll their dice and put, according to the rolled number, the correspondent amount of counters in their respective blocks of five. The player with the higher amount of counters is allowed to take the difference of counters (the one he or she has more) and puts them on the ten-frame (see Figure 1). Afterwards, the blocks are cleared and the dice are rolled again. The player who completely fills the ten-frame first wins the game.
Figure 1: Rolling and comparing

At the heart of the playing environment are the comparison of two numbers and the determination of the difference. Children get the chance to gain insights into relations of differences of two numbers.

(a) There exist different number pairs with the same difference, e.g. 4 and 1, 5 and 2 both have the difference 3

(b) Number pairs with the same difference are characterized by a compensation relationship: \(5-2 = (5-1) - (2-1) = 4-1\).

(c) The difference increases (decreases) if the minuend is increased (decreased) or the subtrahend is decreased (increased).

Meanwhile, children collect, structure and determine the number of counters on their ten-frame and gain experiences in composing and decomposing of numbers.

The learning environment in primary school picks up on the known materials and rules and therefore on the already gained mathematical experiences in kindergarten. An iconic-symbolic form of documentation (recording sheet, see Figure 1) with the possibility for operative variations of numbers complements the learning environment in primary school. Children not only put their counters in their block of five, but record their rolled numbers with crosses (every child in his or her color), determine the difference and write the numerals. These recordings can be ordered and added by equal differences or other criteria. Finally, children can create (without rolling) number pairs with a given difference, e.g. »3-more«. They get the chance to explore the effect on the difference by increasing and decreasing the two compared numbers by the same amount and gain insights into compensation, for example.

Reconstruction of early mathematical understanding processes

To reconstruct interactive processes of understanding, a qualitative approach is chosen oriented at the interpretative classroom research (cf. Krummheuer, 2000). Therefore, mathematical understanding processes are interpreted step-by-step: At first, video episodes are selected and transcribed. In a turn-by-turn process, a group of researchers of mathematics didactics paraphrases the episode and works out plausible
interpretations. To advance the validity of interpretations, a consensus is sought between interprets. For the process of interpretation to be intersubjectively checked in the following section, the relevant episodes are presented, although being aware that even the selection of episodes is an act of interpretation [1]. In a further step, these hypothesized interpretations are developed and reviewed and theoretical elements are gathered. These theoretical elements are also reviewed and extended by comparing with further episodes. The comparison of diverse interactive episodes increases the chance of determining the specific element of a single episode. Thus, the empirical evidence for the theoretical elements enhances and raises them beyond case studies (cf. Krummheuer, 2000). In this way, general findings of the particular case can be provided and local theories can be developed. Hence, mathematical understanding processes can be specified on different levels of mathematical development (e.g. Krajewski & Schneider, 2009). Essential for the mathematical analyses is the epistemological triangle, as described by Steinbring (2005). This analysis tool enables identifying specific reference contexts children use by construing and constructing relations. The epistemological analysis focuses in particular on the reconstructions of the interactive process of constructing knowledge on the basis of actions and interactions.

**EARLY UNDERSTANDING OF MATHEMATICAL RELATIONS: TWO EXEMPLARY EPISODES OF »WHO HAS MORE?«**

With the following two interaction episodes from kindergarten and primary school, different accesses to the understanding of elementary mathematical relationships can be worked out.

**»Who has more?« in Kindergarten**

Mahsum und Dalina play »Who has more?« for the first time with their guiding adult (GA) in kindergarten. The rules are cleared by now. At the beginning of the following scene, Mahsum has six and Dalina has seven counters collected on their respective ten-frame (see Figure 2).

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>Mahsum</td>
<td>I’ve got only four missing <em>(takes his die)</em></td>
</tr>
<tr>
<td>2</td>
<td>GA</td>
<td>Well <em>(pointing with her finger at Mahsum)</em>. You only need four more? What about you? <em>(Pointing with her finger at Dalina)</em></td>
</tr>
<tr>
<td>3</td>
<td>Dalina</td>
<td>And I only need three more.</td>
</tr>
<tr>
<td>4</td>
<td>GA</td>
<td>Only three more. Aha.</td>
</tr>
<tr>
<td>5</td>
<td>Mahsum</td>
<td><em>(rolls number 4, takes four counters while counting)</em> One, two, three, four.</td>
</tr>
<tr>
<td>6</td>
<td>Dalina</td>
<td><em>(rolls number 1, rolls immediately number 1 again and the third time she rolls number 3, takes three counters and starts putting them in her ten frame)</em></td>
</tr>
</tbody>
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*Figure 2: Ten-frames*

1. The whole German transcripts are available from the authors.
Stop.

Dalina: Oh. (Removes the counters and puts them in her block of five)

Mahsum: (meanwhile) I’ve got only one more.

Dalina: (moves her block of five to Mahsum’s) Mahsum has got one more. (…) May I put mine anyway?

In the first lines (1-4), the children point out how many counters are still missing on their ten-frames. The term »only« indicates a comparison. Comparing the filled with the unfilled fields or comparing the unfilled before with the unfilled now, there are less unfilled, so »only four«. Dalina could also have referred to Mahsum’s missing counters. Compared to his four, she only needed three more. In the following, the children roll the dice again. Mahsum rolls number 4 and transfers it into the correspondent amount of counters in his block of five. But Dalina seems not to be satisfied with her number 1. She rolls the die unnoticed for several times until she has number 3. Two interpretations are possible why Dalina chooses number 3: Either she thinks number 3 is better than 1 or she thinks number 3 is best to win. It argues for the latter that she starts to put three counters on her ten-frame (line 6).

After Dalina put her counters in her block of five, it is unquestionable that Mahsum has got one counter more he is allowed to put on his ten-frame (line 9-10). Even before all counters have been put in the block, Mahsum anticipates that it had to be only one more. But the constraint »only« indicates that Mahsum had another expectation after his roll: The number 4 is the second highest possible. Despite the high number, he achieves the smallest gain: 4-3=1. However, it is conceivable to have three better results (4-2=2, 4-1=3, 4-0=4) and only two worse (4-5=-1, 4-4=0).

In line 10 it becomes clear that Dalina put her counters not only mistakenly on her ten-frame (line 6), but considered them to be proper for the unfilled fields of her ten-frame (7+3=10). In doing so she relates the gaps to the rolled number directly although she knows how to determine differences between two amounts of counters in the blocks of five. By comparing these two amounts she experiences that the rolled number 3 compared to 4 does not help to win three counters and fill the gaps.

The guiding adult denies Dalina’s question. The children clear their blocks and roll the dice again.

Mahsum: (rolls number 2) Two (takes two counters and puts them in his block of five)

Dalina: (rolls number 4, looks at Mahsum’s block of five, turns the die to number 5 and raises her hands) Five.

Mahsum again directly transfers the number 2 in the amount of counters in the block of five (line 11). Dalina first rolls number 4, but again changes the die this time without rolling into 5 (line 2). By raising her arms saying »five« she could either detract that she infringed the rules or express that she had won the game.

Due to her endeavor to achieve three counters in the first scene it can be assumed that here, as well, she tries to achieve three counters. Possibly this time she first mentally
compares her rolled number 4 with the 2 counters in Mahsum’s block and determines the difference 2. She realizes that the difference does not suffice to fill the gaps on her ten-frame. Through increasing the rolled number from one to 5, the difference increases by the same amount to 3. With the difference 3 Dalina could win the game, but this time it is noticed that she turned the die and she has to reroll correctly. Compared to the first scene Dalina increases her rolled number in order to achieve a greater difference instead of achieving the equal number according to the number of gaps. It seems that her view on the 3 needed counters changed and she tries to construe them as a difference of two numbers. Nevertheless, it is unlikely that she is completely aware of the relationship: \((4+1) - 2 = 2 + 1 = 3\).

**Summary:** Within the playing environment »Who has more?« children are encouraged to determine numbers and relate them to each other especially by comparing. Comparisons are made in a qualitative way (e.g. »only«) and in a quantitative way by determining the difference. Also due to adaptive guiding, children discover a variety of number relations, even rudimentary operative relations.

1. Determine numbers: Different types of numbers are determined (rolled numbers, amount of counters, filled and unfilled fields on the ten-frame and the block of five).

2. Compare numbers: (a) Differences between two linear structured amounts are determined. (b) Rolled numbers are compared directly. (c) The difference is related and compared to the number of unfilled fields. (d) The rolled number is related and compared to the number of unfilled fields. (e) The filled and unfilled fields are compared. (f) Diverse real and desired differences are compared qualitatively.

3. Decompose numbers: The difference between two numbers is seen as a part from the bigger number and is removed.

4. Compose numbers: (a) The number of new won counters is added to previous counters on the ten-frame. (b) The sum of the filled and unfilled fields is the whole number of fields of the ten-frame.

5. Increase and decrease numbers: greater or specific rolled numbers are desired to take advantage: (a) The rolled number is increased so the amount of gaps and the rolled number are equal. (b) The rolled number is increased to achieve a greater or a certain difference.

This scene shows – although the children are acquainted to construe and determine differences between two linearly represented amounts of counters – that it is challenging to reason the required counters in terms of a certain relation between two numbers and create differences by finding two appropriate number pairs. The first access usually is to desire a rolled number equal to the amount of gaps. But the experience during the game shows that this only works if the subtrahend is zero. Dalina even changes her rolled number in order to acquire a certain gain. While this operative activity infringes the rules of the game in kindergarten, in primary school, the intention of the lesson is that the students create diverse »3-more results«.
In primary school in the third unit, children are presented a fictive score (a ten-frame with seven blue counters and a ten-frame with nine red counters) and are asked to find different possibilities for the blue player to win. The task is to »Find 3-more results«. Children are given recording sheets with two rows up to 10 instead of blocks to record their findings (see Figure 3).

Annabelle and Rene solve this task together, Rene marks the blue, Annabelle the red crosses. On the other side of the table, another pair of students works together. Annabelle marks the determined number 2 of crosses with her red color as Rene comments the solution recording of the pair working vis-à-vis:

1 Rene Oh, you’ve done something wrong. Blue has to win always three (shows three fingers, bends over the table) Oh look, there are only two winning (pointing on the two blue crosses above). Oh!

2 Students Oh, we need a rubber. Do you have a rubber?

3 Rene Look here, look here you can make one more here and circle as well (pointing on the sixth field above the blue crosses and circling with his finger around the 4. to 6. field on the left side). It’s also possible.

This scene exemplifies that the given objective »3-more« causes also differing solutions, which Rene marks as wrong (line 1). Rene emphasizes that the difference always has to be 3 and criticizes that the current difference is »only two«. He creates a relationship between the current and the given difference and distinguishes the current to be smaller. The addressed children do not defend their solution, but ask for a rubber. Possibly the process of correction is associated with deleting and remaking (line 2). In the following, Rene formulates another idea, without or with less deleting necessary: If you add a blue cross, the difference increases by 1: \((5+1)-3 = 2+1\). Instead of deleting, it »is also possible« to add crosses in order to achieve a desired difference (line 3).

The two children are not convinced of this idea. Annabelle lends her rubber provided that she is allowed to erase by herself. She removes the third red cross, thus benefits from the mathematical relation that the difference can also be increased by removing a red cross: \(5-(3-1) = 2+1\).

Meanwhile Rene attends to the next recording sheet:

4 Rene (marks six fields with blue crosses, points on the third field of the left row and adds four further blue crosses) I’m rolling ten.

5 Annabelle Ten? We aren’t allowed here (pointing with her pen to the upper fields)
6 Rene    Sure (.) we are.
7 Annabelle What am I rolling now?
8 Rene    You’re rolling (.) ehmm eight (pointing on the seventh field of the right row)
9 Annabelle Okay (marks the fields beginning bottom up by counting) One, two, three, four, five, six, seven
10 Rene    Seven (points at her red crosses and starts to mark a further cross).
11 Annabelle No. No (.) Or else I’ll lose. Look, I’ll win with this (pointing to the three upper blue crosses) and here (pointing to the right numeral field below the crosses) you have to write seven.
12 Rene    Stop.

Rene initially marks six crosses and finds a preliminary solution (the number pair 6 and 3) by pointing on the third field on the left side. This would have been exactly the solution he had offered to his classmates. Maybe because Annabelle is still rubbing or because this solution is not clear yet, Rene adds four further blue crosses (line 4). With these ten crosses he exceeds the fifth field for the first time. This possibly causes a short confusion and leads to Annabelle's objection (line 5-6). Rene answers Annabelle’s question that she had to roll eight. Possibly he miscounted because at the same time he points on the seventh field auf Annabelle’s row (line 7-8). But now Annabelle wants to mark the »rolled number« 8 correctly. First she does not comply with the interruption by Rene after seven crosses (line 11). Just at that moment as Annabelle marks the seventh cross, Rene seems to recognize that the desired difference of 3 is reached respectively the field he had pointed before. To conserve this difference of 3 has now priority over marking the predefined number. One reason for avoiding further red crosses in any case might be that he could hardly add blue crosses to repair the difference. Instead, he tells Annabelle to write the numeral 7 in the corresponding field (line 12). Rene argues that in this way he would win otherwise he would lose. For the difference 3 not the previous determined number is crucial, but the smaller number of crosses ends three fields below the bigger: a-(a-3) = 3.

Summary: The analyzed scene shows how children in first class try to represent the number 3 as difference between two numbers. In doing so, they explore different solutions (number pairs) for the difference 3. These solutions are regarded initially isolated. The correction of the preliminary solution 5-3=2 demonstrates how relationships between numbers can be used to create or change differences: If the difference should be increased by one, either the minuend has to be increased by one or the subtrahend has to be decreased by one: (a+1)-b = a-(b-1) = (a-b)+1. At the same time, the children only mention the difference and the necessary changes, the single numbers to construct the difference are not relevant. In the second section a more general way of interpreting the difference of 3 can be reconstructed in the sense of three overhanging crosses. In this view, the relation »3 more« is independent of two concrete and certain numbers, but applies for all number pairs which meet this relation.
(1) Determine numbers: (a) Numbers are determined to specify both the number on the imaginary dice and the numeral that has to be written. (b) The amount of the difference is determined.

(2) Compare numbers: Numbers are compared to determine if the correct difference of 3 is constructed. By comparing with the desired difference of 3, differing differences can be identified.

(3) Decompose numbers: The difference is seen as a part of the minuend. In order to create number pairs with the difference 3, the subtrahend has to be smaller by 3, so three fields have to be left empty i.e. three crosses have to overhang.

(4) Compose numbers: After increasing the minuend, the new difference is the sum of the increased amount and the old difference.

(5) Increase and decrease numbers: (a) Increasing the minuend in order to increase the difference by the same amount (b) Decreasing the subtrahend in order to increase the difference (c) To every number x (≥ a), which should be greater by a corresponding number can be found to have the property to be smaller by a, i.e. x-a.

COMPARISON OF THE LEARNING SITUATION IN KINDERGARTEN AND PRIMARY SCHOOL AND OUTLOOK

Both presented and interpreted scenes of kindergarten and primary school indicate that dealing with complementary learning environments offers children space for mathematical experience and space for mathematical play. Both learning situations, the game in kindergarten as well as the more systematic formal learning in primary school, create room for exploring and discussing elementary relationships of differences. In the following, the observed learning situations are compared with regard to the process of playing and the process of exploring and using number relations.

Playing in kindergarten and primary school: In kindergarten, children get acquainted with »Who has more?« as a game which can be played repeatedly with other children and with a guiding adult. From the process and the aim of playing occasions arise for interpreting numbers in relation to other numbers and for discussing them with others. Playing the game offers children rich space for mathematical experiences in order to construct number relations especially to interpret differences (»Mathematics embedded in Play«, Ginsburg, 2006). In primary school, however, the task »Find 3-more-results« opens space for mathematical play. This is particularly evident in the identified context »increase and decrease numbers«: children take advantage of the opportunity to vary the amount of crosses, choose bigger or smaller amounts or change amounts and explore the effects. Herein »Play Centering on Mathematics« (Ginsburg, 2006) is expressed.

Exploring and using number relations: Children in kindergarten learn a variety of ways and strategies to determine, compare, compose and decompose numbers; i.e. they explore rich number relations by dealing with the material (cf. Stebler et al.). In primary school, one special mathematical aspect is focused: the relation of differ-
ences. But here, too, rich number relations can be explored and used. For example, the conception of the difference as a kind of »entity« is developed, which is included in the minuend (as a part), but is independent of concrete numbers. Further systematic exploration and experience of number pairs with the difference 3 can children lead to deeper insights in compensation: \( a-b = (a+x)-(b+x) \).

**Complementary learning environment in transition:** The analysis shows that each learning situation represents specific institutional characteristics in terms of play and number relation. But at the same time they are mutually linked: So already in kindergarten arises the potential for mathematical play with differences, which becomes more differentiated by the focused mathematical activity in primary school.

In view of the activity in primary school, children construe differences in the (game) material in kindergarten. The confident dealing with the representation and the language of the children (»roll«, »I win«, ...) in primary school indicate the use of the gaming experience. The game gains in relevance as a meaningful and motivating context for mathematical activities (cf. van Oers 2010). The experience of construing differences in the material can be used in order to create and record number pairs of a given difference up to 10, i.e. to construct differences. For that purpose the materials have to be reinterpreted: The recording sheets are no longer primarily used to construe differences, but rather to construct differences. This can also be seen in the term »rolling« used by the children: Roll the dice no longer means to generate number pairs randomly, but rather to find compatible number pairs by themselves. Both the »rolled« and the recorded number can be increased and decreased accordingly. Thus, the space for mathematical play enhances. By means of complementary learning environments based on mathematical-rich games with structured game materials, playful mathematical handling can be stimulated by reinterpretation of the materials as mathematical objects. So the particular interactive learning situation provides incidental mathematical experience on the one hand and deeper insights in relations between numbers on the other hand. In this regard they are complementary.

The focus of this paper is the children’s process of understanding differences at the end of kindergarten and at the beginning of primary school. The highlighted disparities of playful mathematics in games and mathematical play can also be seen in other playing and learning environments of the project. The approach to enable children in kindergarten rich experiences via mathematical games, which will be taken up and continued in primary school in form of »mathematical play« poses a promising way to arrange linked academic learning processes in transition from kindergarten to primary school.

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