DEVELOPMENT OF A FLEXIBLE UNDERSTANDING OF PLACE VALUE

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In this paper we highlight the importance not only of an understanding of place value, but the importance of a flexible one. We describe the principles of our decimal place value system and the development process of the children. Embedded in the Artefact-Centric Activity Theory we present an education-oriented design of a virtual place value chart and its potential to support this development and understanding of a flexible place value. In the following article we highlight some results of a quantitative study with 3rd-graders that guides our further research in that area.

INTRODUCTION

The understanding of place value and in particular the *flexible* understanding of place value plays an important role in learning and understanding mathematics. We define the flexible understanding of place value as the ability to switch between different possibilities to split a whole in parts whereupon the parts are multiples of powers of ten, e.g. 19 hundreds 77 ones is the same as 1 thousand 9 hundreds 7 tens 7 ones is the same as … A flexible understanding of place value is not only the basis of understanding the written calculation methods of addition, subtraction, multiplication and division but it is also important for calculation strategies or to get along in everyday life. In that way we need to know e.g. that 3 hundreds 14 tens 7 ones is the same as (3+1) hundreds 4 tens 7 ones (see Figure 1) to understand what the 'little ones' (the carry over) mean.

	H	*T*	*O*
	1	*7*	*2*
+	*2*	*7*	*5*
	3	*14*	*7*
	4	*4*	*7*

Figure 1: written calculation method

And with a flexible understanding of place value we can easily divide e.g. 361 218 by 6 without the written algorithm because 36 ten thousands 12 hundreds 18 ones divided by 6 is the same as 6 ten thousands 2 hundreds 3 ones that is 60 203. According to this, the flexible understanding of place value is needed to understand polynomial division in later years. But we do not only need it to calculate, there are nonstandard partitionings all around us. E.g. we have to travel fourteen hundred kilometres (14 hundreds = 1400), the trip costs twelve hundred Dollars (12 hundreds = 1200) in the year nineteen hundred seventy two (19 hundreds 7 tens 2 ones = 1972). There are a lot of situations where we do not use the standard form of representation but a flexible one. That is why it is important for the children to understand and to be able to switch between different forms of representation of one and the same number.

In the following section we describe how children develop the decimal part-whole concept that is the basis for our decimal number system. We then explicate the principles of our decimal number system.

DEVELOPMENT AND PRINCIPLES OF THE DECIMAL NUMBER SYSTEM

The children already develop a general part-whole concept before they acquire the decimal part-whole concept (Resnick 1991, Ladel/Kortenkamp 2011, 2014). The children experience that all numbers are additive compositions of other numbers that is a conceptual base of elementary arithmetic. “This compositional character of numbers provides an intuitive basis for understanding fundamental properties of the number system.” (Resnick 1991, p. 375) The compositional character of numbers is already preparing the understanding of our number system that follows the *additive property* (1). This principle means that the quantity represented by the whole numeral is the sum of the values represented by the individual digits (Ross 1989, p. 47). But in this stage of development it is very general.

(I) $P_1 + P_2 + \ldots + P_k = W$

Based on such general parts the teacher has to instruct the children to make special parts, namely multiples of powers of ten. This is because of the *base-ten property* (3) of our number system that means that the values of the positions increase in powers of ten from right to left. To get the value of an individual digit we have to multiply the face value of the digit with the value assigned to its position that is the *multiplicative property* (4).

(II) $n_k \bullet 10^k + n_{k-1} \bullet 10^{k-1} + \ldots n_o \bullet 10^0 = W$

To create these parts the children have to make bundles. In our decimal number system only bundling is not enough. We have to continue the bundling until it is not possible any more (*principle of continued bundling* (2)). In doing so all parts (n_i) are single-digit ($n_i<10$). The teacher has to admonish the children to bundle till the end.

(III) $n_k \bullet 10^k + n_{k-1} \bullet 10^{k-1} + \ldots n_o \bullet 10^0 = W$, and $n_i<10$ for all i.

In this way the understanding of bundling is a necessary condition for understanding place value. Nevertheless the understanding of place value and its acquisition has to be distinguished!

When learning numbers and place value the children get in contact with different forms of number representations. We distinguish

- numbers indicated by bundle units, e.g. 4 ones 23 tens
- numerals, e.g. nineteen hundred seventy seven

- points in a place value chart
- numbers in a place value chart
- numbers.

The constraint that the parts in (III) have to be single-digit is a specification that is only needed if numbers are represented in the last mentioned form, e.g. 1971. In all the other forms of representation the parts (n_k) can also be higher than 9 (see (II)).

Only in numbers we need the *positional property* (5) where the quantities represented by the individual digits are determined by the position they hold in the whole numeral and not by the indication of the bundle units, not by the indication words like "-teen", not by the column and its designation.

As we defined earlier in this article the flexible understanding of place value is the ability to switch between different possibilities to split a whole into multiples of powers of ten (II and III), in several forms of number representation. Concerning the different kinds of partitioning we distinguish standard partitions (III) and nonstandard partitions (II). The nonstandard partitions we distinguish again in strong and not strong nonstandard partitions (see Figure 2). A strong nonstandard partitioning can easily be transferred in a standard partitioning because bundle-neighboured quantities just have to be summarized, e.g. 2 hundreds 31 ones to 231.

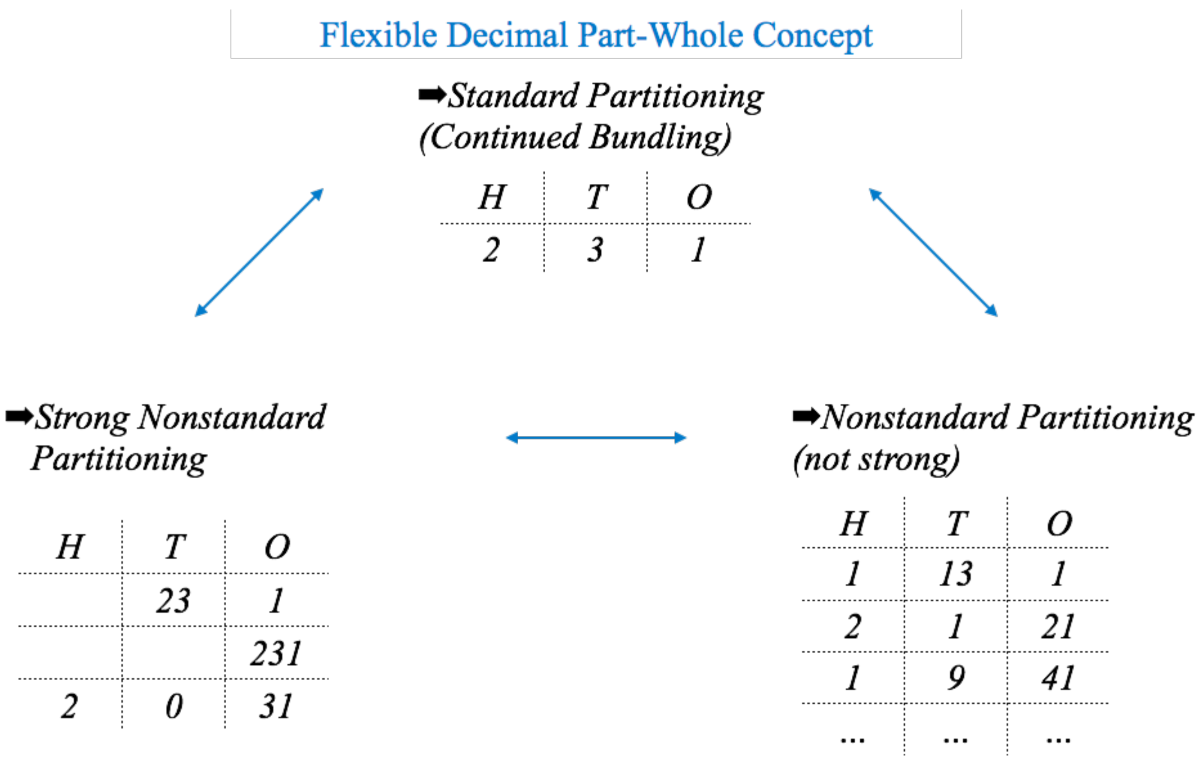


figure 2: flexible decimal part-whole concept

THE PLACE VALUE CHART AND THE MEANINGS OF ACTIONS

There are different possibilities of meaning that an action with an artefact may have. Bartolini calls it the *semiotic potential* of the artefact and refers to the triangle artefact, task and mathematics knowledge. There is “a double semiotic link between the artefact and a task on the one hand and the artefact and mathematical meanings on the other hand. The former is within the reach of students whilst the latter emerges from the epistemological analysis made by teachers and experts.” (Bartolini 2011, p. 96) They are the teachers and experts who put their knowledge about mathematics in an artefact. According to that an artefact can ‘behave’ or externalize in different ways when actions are done with it.

In the place value chart there is the one action of moving a token, but there are different mathematical meanings. With real material moving a token in the place value chart often means a change of value, e.g. one token in the tens becomes one token in the ones, so the change of value is -10 + 1 = -9. Another meaning of moving a token is a change of representation, e.g. one token in the tens becomes ten tokens in the ones that means an unbundling with constant value.

In that way the children internalize different mathematical knowledge about the meaning of moving a token in the place value chart depending on the externalization of the artefact (see Figure 3).

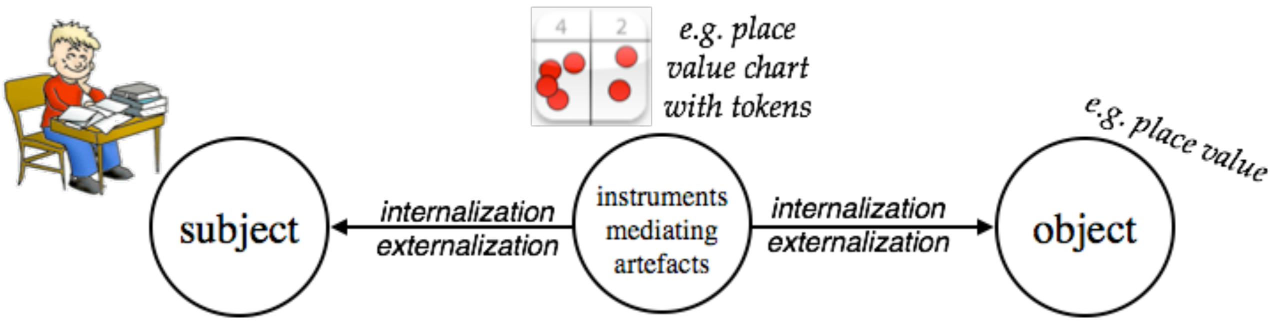


Figure 3: Main axes of the Artefact-Centric Activity Theory

There is no ‘right’ and no ‘wrong’ meaning because both of them do have their justified existence. But the different behaviours of the token and externalizations of the artefact focus on different principles of our number system. Whereas a change of value mediates the children that if the position of a number changes its value changes too, the constant value mediates the children that if the position of a number changes its value have to be multiplied or divided by multiples of the power of ten so that the value is constant.

RESEARCH

The fact that all kinds of number representations and all kinds of partitioning, standard and non-standard, strong and not strong, have been found in schoolbooks (see Table 1) emphasises the importance of being concerned with them. In most tasks the children have to change the form of representation of the number and in doing so they have to transfer a not strong partitioning in a strong one.

Number representation	**Example**
numbers indicated by bundle units -> numbers in a place value chart	Trage ein. l) 8E 5H 4T 6Z — ZT T H Z E
numbers indicated by bundle units -> numbers	Wie heißen die Zahlen? 3H 16Z 8E = ____
numeral -> numbers	Lies die Zahlen und schreibe sie mit Ziffern. vierundzwanzigtausend neunhundertachtzehn
points in the place value chart -> numbers	Schreibe die Zahlen mit Ziffern und lies die Zahlen. a) M HT ZT T H Z E — 2a) 1 2 3 0 1 5

We arranged a quantitative study with 255 3rd-graders resident in Halle and Saarbrücken (Germany) and Luxembourg as well as a qualitative study with 52 children at the end of 2nd class. We will only report on the quantitative study here, but the qualitative data has been used for the design of the study.

QUANTITATIVE STUDY

For the quantitative study, we created a 30 min test of three parts. When administering the test, the children were allowed to work for 10 minutes on each part.

The first part consists of two tasks of different types. The first task is to compare two numbers each given as certain numbers of hundreds (H), tens (Z, German "*Zehner"*) and ones (E, German "*Einer*"). The children should circle the number that is larger. For completeness, we list all eight subtasks:

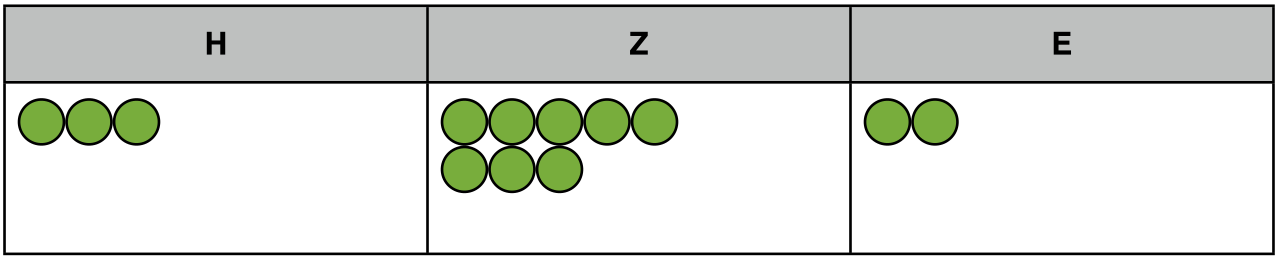
1a)	2H 5Z 3E	2H 4Z 17E	1e)	34Z 14E	3H 7Z 3E
1b)	3H 13Z 5E	4H 6Z 5E	1f)	2H 1Z 15E	21Z 4E
1c)	735E	7E 3Z 5H	1g)	7H 3E 6Z	7H 6Z 1E
1d)	1H 43Z 9E	63Z 9E	1h)	8H	91Z 3E

Note that both strong and non-strong partitions are used and the order of hundreds, tens and ones differs between subtasks.

The second task is to write a number given in the notation shown above into standard notation. This again involves both re-ordering and simple calculations when non-strong partitions are used:

2a) 3H 6Z 1E	2c) 1H 32Z 4E	2e) 7H 3Z	2g) 7E 31Z
2b) 3Z 23E	2d) 3E 2H 5Z	2f) 279E	2h) 3H 3Z 14E

The second part again consists of two different tasks. The first task is to write down a number that is shown in a place value chart. We only show the image of the first number (3H 8Z 2E) of task 1a:



Three other representations are shown with the numbers 1H 2Z 13E (Task 1b), 2H 12Z 13E (Task 1c), and 5H 15Z 3E (Task 1d).

The second task of the second part is the inverse task to the first one: Children should draw tokens into an empty place value chart for the numbers 314, 163, 542 and 304 (Tasks 2a–2d of Part 2). In addition, we ask them whether they can think of a different way to represent the number in the place value chart (German: “Kannst du die Zahl auch anders in der Stellenwerttafel darstellen?”). No further explanation of what “different” means was given.

The third part of the test repeats the questions of the first part with different numbers:

1a)	1H 7Z 2E	1H 8Z 15E	1e)	51Z 23E	4H 3Z 8E
1b)	3H 25Z 7E	2H 9Z 4E	1f)	3H 4Z 27E	45Z 6E
1c)	385E	3E 8Z 5H	1g)	8H 1E 3Z	8H 3Z 1E
1d)	2H 27Z 8E	53Z 7E	1h)	7H	84Z 2E

2a) 2H 8Z 3E	2c) 2H 41Z 5E	2e) 8H 1Z	2g) 5E 53Z
2b) 4Z 51E	2d) 2E 3H 7Z	2f) 329E	2h) 6H 2Z 35E

About half of the children had access to an iPad with an interactive place value chart (Ladel & Kortenkamp, 2013) while working on part 2. No further instruction besides basic usage of the place value chart app (adding, deleting and moving tokens) was given, and it was the first time the children had contact with that app. In our data we recorded whether the children had access to the iPad, but we did not monitor how the children used it, or whether they used it at all.

For the analysis, we marked the tasks in parts 1 and 3 as well as in part 2, task 1, for correctness. The inverse operation of representing numbers in the place value chart was analysed by categorizing the answers for the first subtask (“represent 314 in the place value chart” and “can you find a different representation”). The answers for the three other numbers usually did not fall into another category, but it happened that they were left unanswered due to time or other constraints. We found 9 different categories:

1. *Flexible Answer (N=56)*: Students who gave three representations of 314 that were correct and used different (nonstandard) partitions. Note that only a small percentage of students gave only two representations.
2. *Flexible Answer with errors (N=31):* Students who gave three representations of 314 using different (nonstandard) partitions, but made minor errors that can be attributed to wrong counting or miscalculations.

3. *Base-Ten-Block Errors (German: “Mehrsystemfehler”, N=19)*: Students who made mistakes that can be explained by mixing base-ten-blocks and the place value concept, for example putting 10 tokens into the “tens” cell to represent 10.
4. *Other symbols (N=24)*: Students who used other symbols like flowers instead of circles when giving a “different” representation.
5. *Permutation (N=69)*: Students who just permutated the 3, 1, and 4, thus producing wrong representations showing for example 413 or 341.
6. *Only one Representation (N=16)*: Students who just gave the standard representation.
7. *Other arrangement (N=5)*: Students who used the same symbol (a circle) for other representations, but only changed the arrangement within the cell, not the number.
8. *Value changing (N=4)*: Students who used the same number of tokens (3+1+4=8), but moved them into other cells, thus creating representations of other numbers.
9. *Non categorizable (N=26)*: Students who gave other representations that could not be categorized at all. Representations were showing other numbers with other token counts.

In addition, N=5 students did not answer at all, giving a total of 255 answers.

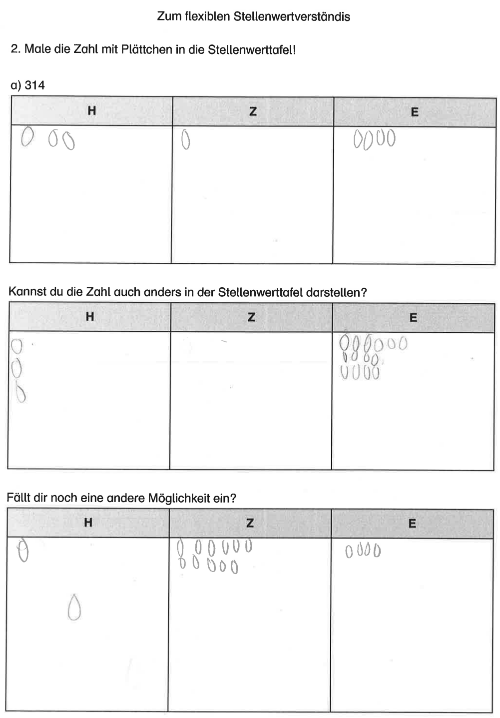


Figure 2 A flexible answer in part 2

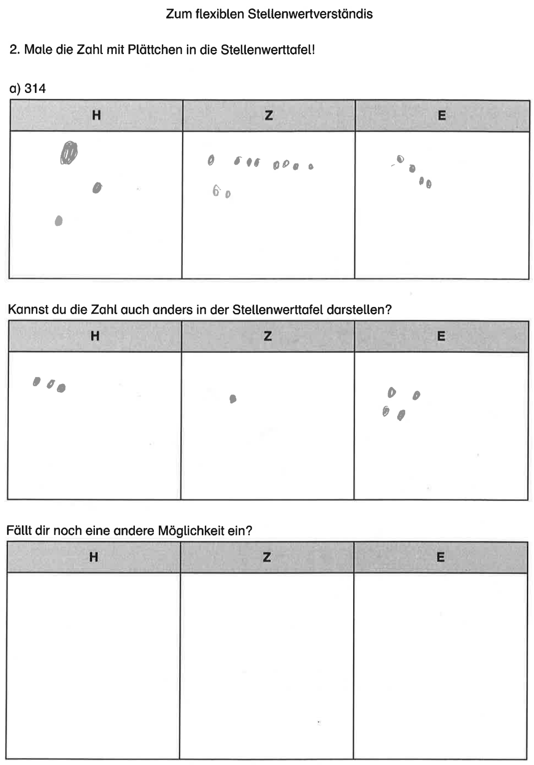


Figure 3 A Base-Ten-Block error of a student

ANALYSIS OF THE QUANTITATIVE DATA

We used Statistical Implicative Analysis (SIA, Gras et al., 2008) to analyse the data collected. With SIA, we can find implications between (binary) variables. The algorithm calculates an implication intensity between two variables that measures the measures the "surprisingness to observe a small number of counter-examples." (Gras et al., 2008, p.16) The intensity is a number between 0 and 1, where 1 means that it is no surprise at all to see a small number of counter-examples, corresponding to the fact that we suspect an implication between these two variables. Note that we are indeed looking at implications, as opposed to mere correlations.

In Fig 4 you see the dependency graph created by our own analysis tool (Ruby code available on request from the authors). Input were the binary variables for correct and incorrect solutions of subtasks (TxAyz is true when Task yz in Part x was solved correctly, NTxAyz is true when Task yz in Part x was solved incorrectly or not at all), the binary variables APP and NAPP that are true when the students had access to the iPad app in part 2 resp. did not have access to it, and binary variables for each of the 10 categorizations (including "no answer") of the answer for Task 2a in Part 2. Category variables are marked in yellow, correct answer variables in green, and wrong answer variables in red. The NAPP variable is shown in white. Implication intensities between variables are shown by arrows labelled with the values rounded to .5%. We removed transitive implications for better readability of the graph.

We only show implication intensities that reached a level of 95%, with the exception of those that are connected to APP or NAPP, where we show all three implications found that have an intensity larger than 90%. You will not find APP in the graph, as it was not connected to any other variable with an intensity of more than 90%.

A first glance immediately shows that –unsurprisingly– the "correct answer" items and the "incorrect answer" items are highly connected, i.e. there is a high internal consistency of our test. The only two outliers are NT2A1a → T1A2g and NT1A2g → T2A1a, both with a very high intensity of 99.5%. Recall the tasks T2A1a and T1A2g: In T2A1a the 382 in standard partitioning should be read off a place value chart, which is a very easy and basic task. In T1A2g children are asked to translate 7E 31Z into 317, which includes both non-standard partitioning and the permutation of places as additional hurdles. The data shows that students who fail at the easier (actually, the easiest in this test) task are likely to master the harder one, and vice versa, students who fail at the harder task master the easy one. Our interpretation of this surprising fact is that better performing students might underestimate the easy task, a common problem of gifted students. That students who fail at the harder task master the easier one can be explained by the fact that the easier task can be solved by all students (but the gifted ones).

Apart from this internal consistency, we see also that there is no way to conclude the students' category from their answers in parts 1, 3, and the first task of part 2, as the implication intensity of implications towards the category variables are below the threshold. Just looking at the correctness of answers in the easy-to-check tasks is not enough to diagnose the student. The categories “Other Arrangement”, “Value Changing” and “Non Categorizable”, as well as “No Answer” are not connected by presumed implications at all, which can easily be attributed to the low numbers of cases and the non-categorizability.

On the other hand, we can find presumed implications originating at the categories towards the correct answers or incorrect answers cluster. With high implication intensities of 96%, 99%, 98%, 97% and 98.5% the “flexible thinkers” solved T2A1a, T2A1d, T1A2d, T3A2h and T3A2d correctly. Recall that we removed implications that are implied due to transitivity from the graph, so these are just the “entry points” into the cluster of correct answers. Even students in the “flexible with errors” category end up solving T1A2e and T1A1b correctly, albeit with lower implication intensities of 97.5% and 95.5%. Still, there is no above-threshold implication into the cluster of wrong answers for any of the flexible categories.

This is different for the “Base-Ten-Block error” category: With implication intensities of 98% those students give wrong answers in T3A2g and T3A2c. There is also a presumed implication into the correct answer block, but at lower intensities of 96.5% (T1A1e) and 95.5% (T1A2g). It is safe to say that students who put 10 tokens into the tens column to represent 10 are more likely to solve the tasks incorrectly.

Students in the “Other symbol” category show implications both to the “correct answers” cluster and, very weakly, to the “incorrect answers” cluster. The latter implication is only via the NAPP variable, which is shown as implication of “Other symbol” with 95.5% intensity. As we chose the children who were working with the app and those who did not by random, we cannot draw conclusions here. In fact, the wording of the question (“Can you show a different way to represent that number”) is very open and might be interpreted in various ways by students. In those cases only a qualitative design could give more insight.

Finally, it is likely belonging to the categories “Permutation” (95.5% to NT2A1b, 96.5% to T3A2b) and “Only One Representation” (96.5% to NT1A2g) again implies wrong answers in the other tasks.

DISCUSSION

The quantitative data shows that there is a connection between flexible representations in the place value chart and correct solutions of tasks that involve interpretation of numbers given in non-standard representations. Even students who make mistakes when giving several different representations (with “different” being “different partitioning”) are likely to solve the other tasks correctly. It seems to be sufficient to be aware of the fact that numbers have different representations in the place value chart.

Contrary to that, students who have difficulties with representations in the place value chart are more likely to fail at tasks that do not involve the chart itself, but need competences in place value and non-standard partitions. Note again that we analysed implications – nothing is being said about students who solve the tasks successfully, they could lack the place value chart skills or could be flexible thinkers. But our data shows that working with place value charts *and non-standard partitions* can improve students' performance.

We can also see that the transition from sorting base-ten-blocks into place value charts and abstracting from the blocks into tokens for counting is crucial for a proper understanding. Students who mix up "ten" as a quantity and "ten" as a place value showed problems with solving other tasks. We suggest not to sort base-ten-blocks into place value charts at all. Instead, it is a better solution to use a place value chart for counting marks, each mark representing one block of the proper size. This way there is no confusion of the block representing the number 10 (or 100) and the counting token for "a ten" (or "a hundred").

Our test design did not involve a treatment, so we cannot conclude anything about the effect of using the place value app. The APP variable does not appear at all in the graph, the NAPP variable has a very weak connection to the "incorrect answer" cluster. This can be taken as a hint that using the app could improve students' performance, or better, could improve their flexible use of the place value chart, but this must be left for further investigation.

CONCLUSION AND OUTLOOK

We could show that students who show a *flexible* understanding of place value are more likely to solve tasks that involve non-standard partitions of numbers into hundreds, tens and ones. Such tasks appear in textbooks and in everyday life, and are a well a basis for addition and subtraction algorithms.

As a next step we will try to gather more qualitative data related to the quantitative results. For this, a follow-up study with 3^{rd}-graders in Halle is planned, where students are interviewed based on their results in a new quantitative study using the same test as the one presented here.

A text book analysis (not detailed here) has shown that there is a lack of emphasis on the flexible interpretation of place value charts, while there are still tasks in the book that require this understanding. Based on these considerations and the results of this study, we will design activities involving the interactive place value app that could improve students' understanding, and we will test these activities as treatments in a pre-post-design.

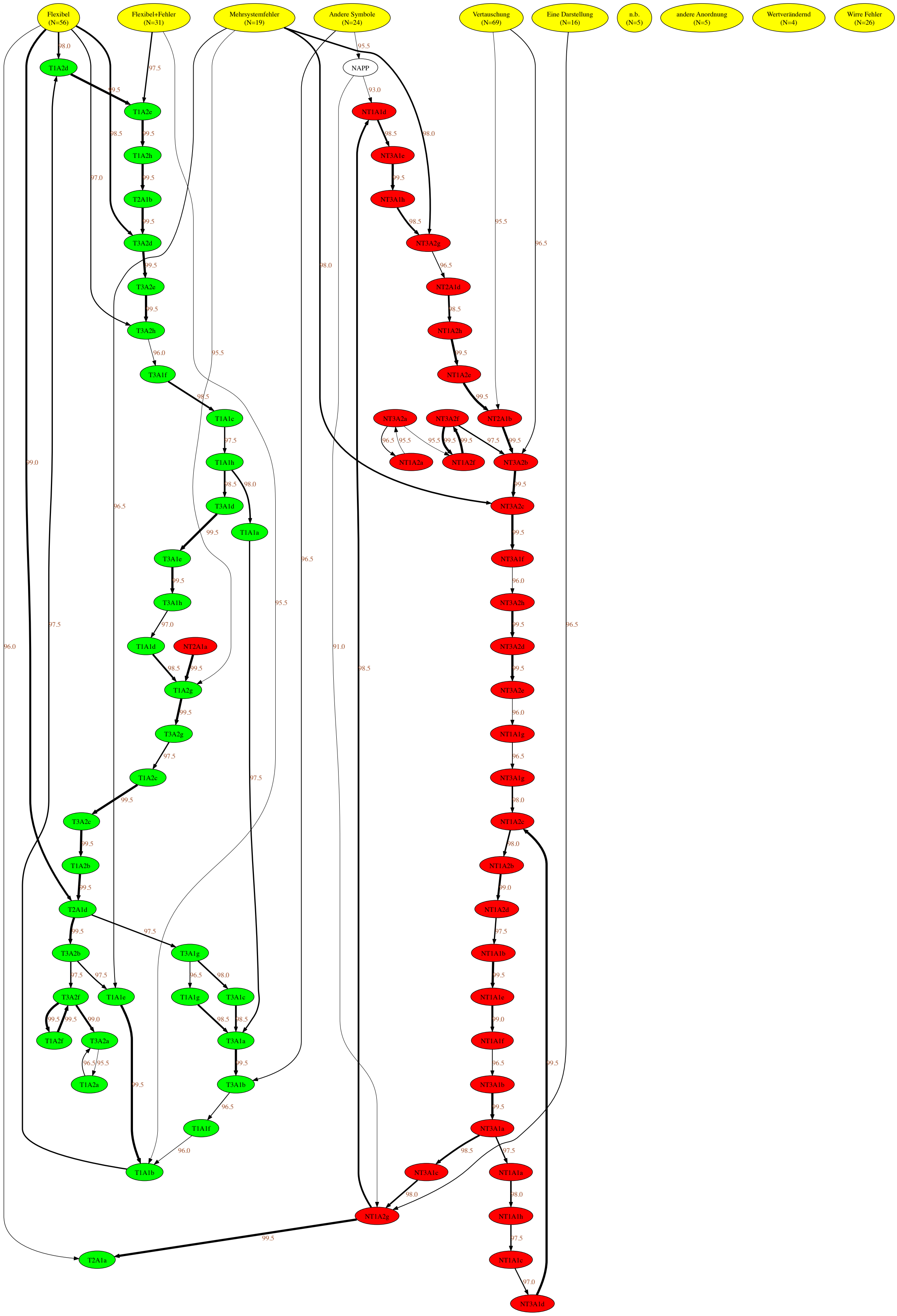


Figure 4 Dependency Graph of the Quantitative Data

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