THE SIGNIFICANCE OF THE EQUAL SIGN IN THE DEVELOPMENT OF EARLY ALGEBRAIC REASONING

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In this presentation, we analyze the process of building algebraic thinking in children as it relates to the necessary conceptualization prerequisites which give meaning to the ideas underpinning the basic rules of algebra (Radford & Grenier, 1996).

In recent years, National Council of Teachers of Mathematics (NTCM) publications and the Ontario Ministry of Education (OME) continue to provide resources toward the development of algebraic reasoning starting in early childhood with a view to examining the equal sign as an indicator of mathematical equivalence and equality. In early childhood, the relationship between equivalence and equality is a key element that integrates different facets of building digital learning (Theis, 2005). Moreover, empirical evidence suggests that young children’s knowledge of the equal sign as an equivalence relationship acts as a key link between arithmetic and algebra and provides insight into the future use of algebraic strategies (Matthews, McEldoon, Rittle-Johnson & Roger, 2012; Alibali, Knuth, McNeil & Stephens, 2006).

We also analyse algebraic thinking by examining mathematics-related tasks completed by twenty-one 5-year old children. Our purpose is to highlight the use of landmark strategies, big ideas and models by Fosnot & Dolk (2010) in regards to the equal sign and their significance in the development of early algebraic reasoning.

INTRODUCTION

This paper is an important part of a research project into the development of algebraic thinking at the pre-school level conducted by researchers from the Université du Québec en Outaouais, Canada. The difficulty surrounding the transition from arithmetic to algebra is well documented and widely recognized (Jacobs, Franke, Carpenter, Levi & Battey, 2007; Knuth, Stephens, McNeil & Alibali, 2006; Falkner, Levi & Carpenter, 1999; Kaput, 1998, 1999). For instance:

During the last decade, more and more mathematics educators suggest initiating the study of algebra at the primary level. They argue that this is not early teaching of algebra at the secondary level nor is it `pre-algebra`` […] , but rather to help students develop algebraic thinking without necessarily using the textual high-level algebra language. (Squalli, 2002, p.4).

While several research articles are aimed at the development of early algebraic reasoning (Cai, & Knuth, 2011; Carpenter, Falkner & Levi, 2000; Saenz-Ludlow & Walgamuth, 1998; Squalli, 2002; Theis, 2005), most of these studies address the
subject at the secondary and primary levels. However, as reported by the Ontario Ministry of Education (OME):

Algebraic reasoning should be promoted and cultivated in kindergarten. We all have the ability to think algebraically, for algebraic reasoning is essentially how humans interact with the world. (OME, 2013, p.3)

In accordance with the research community, “[u]nderstanding and using algebra is dependent on understanding a number of fundamental concepts, one of which is the concept of equality.” (Knuth et al., 2006). According to Jacobs et al.:

Algebra has become a focal point for educators and policymakers, and the widely used catchphrase “Algebra for All” has underscored the importance of providing all students access to algebra [calling it] the gatekeeper to higher mathematics. (Jacobs et al., 2007, p. 258).

Thus, one of the focal areas of work for our team is to study the development of algebraic reasoning in early childhood with a view to examining the equal sign as an indicator of mathematical equivalence and equality.

THE EQUAL SIGN AS AN INDICATOR OF MATHEMATICAL EQUIVALENCE AND EQUALITY

In mathematics, equivalence is defined as any relationship that is reflexive, transitive and symmetrical. In pre-school, a relationship of equivalence is often found in the form of “the same number of elements” therefore a quantitative equivalence. In that particular relationship, two sets of components can have the same number of elements without necessarily consisting of identical objects. On the other hand, the equality is a quantitative equivalence in a specific case where two sets have the same number of elements and these elements are exactly the same (Theis, 2005). Indeed, the expression "2 + 3 = 5" may correspond to a situation of equality where a child assembles sets of elements, for example, a child may start with 2 marbles, win 3 during a game and end up with 5. However, where two children compare the number of marbles, for example, one has 2 in his right hand and 3 in his left, and the other child has 5 marbles, it is a matter of quantitative equivalence because we do not compare the same elements. Nevertheless, the formal expression "2 + 3 = 5" describes two sides of the "=" sign representing exactly the same number, which means, a numerical equality (Theis, 2005). Thus, the symbolic expression of equality alone does not distinguish the type of situation under consideration. As Theis (2005) points out, “an adequate concept of the “=” sign as an indicator of a relationship of equivalence is crucial to understand these properties in the context of different basic arithmetical calculations.
STUDENT’S UNDERSTANDING OF THE EQUAL SIGN

Third graders’ interpretations of equality and the equal symbol:

One concept that is fundamental to algebra understanding and that has received considerable research attention is that of equality and, in particular, understanding of the equal sign (e.g. Alibali, 1999; Behr, Erlwanger, & Nichols, 1980; Falkner, Levi, & Carpenter, 1999; Kieran, 1981; McNeil & Alibali, 2005). The ubiquitous presence of the equal sign at all levels of mathematics highlights its importance. (Knuth et al., 2006, p. 298)

In the Peircean model, by attempting to define and translate an idea, the equal sign is a symbol accomplishing a privileged contact with the object. According to Sáenz-Ludlow and Walgamuth (2007), the equal sign is a good example in history of the complexity involved in interpreting a mathematical concept with a particular symbol and the adoption of symbolic conventions. In accordance with the authors, the historical evolution of this symbol “exemplifies for us that mathematical meanings are not directly conveyed by the symbols without the interpreting activity of the individuals” (Sáenz-Ludlow & Walgamuth, 2007, p. 156). They hold the same true in the classroom as children construct mathematical meanings, in the Peircean model, the interpretant, for this conventional symbol, the equal sign.

According to Carpenter, Falkner & Levi (1999), children need to understand equality as a relationship for two reasons:

   a) The equality relationship is central to arithmetic because children need to think about relationships expressed by number sentences. Children with this ability will be able to represent arithmetic ideas, to communicate them and further reflect on these ideas. A child with the opportunity to express and reflect on the relationship expressed by number sentences might be able to use the same mathematical principle to solve more difficult problems.

   b) The second reason for the lack of understanding about equality as a relationship may be that “one of the major stumbling blocks for students when they move from arithmetic to algebra (Kieran 1981; Matz 1982)” (Carpenter et al., 1999, p. 234). Children who interpret the equal sign as something to do will have to resort to memorizing the rules for solving the equations that could be resolved by manipulating numbers on both sides of the equation if the equal sign had signified a relationship between two expressions. (Carpenter et al., 1999)

These authors continue:

   children as young as kindergarten age may have appropriate understanding of equality relations involving collections of objects but have difficulty relating this understanding to symbolic representations involving the equal sign. (Carpenter et al., 1999, p. 233)
In this regards, they advise that “[t]eachers should also be concerned about children’s conceptions of equality as soon as symbols for representing number operations are introduced. Otherwise, misconceptions about equality can become more firmly entrenched” (Carpenter et al., 1999, p. 233). They show that research has documented evidence of elementary grade children’s misconceptions of the equal sign as it “means that they should carry out the calculation that precedes it and that the number after the equal sign is the answer to the calculation” (Carpenter et al., 1999, p. 233).

Radford (2002) explains that one of the didactic problems in teaching and learning algebra is related to understanding algebraic symbolism and meaning. These errors may be proof of students’ difficulties in understanding the intricate system of rules of sign-use in algebra. As advocated by Vygotsky (1962), Radford, Demers & Miranda (2009) consider that mathematical abstraction is relational in the sense that “we can learn a significant amount of facts, but if we are unable to link them together […] that is, to form a concept linking them, we cannot move to a higher conceptual level.” (Radford et al., p. 11). In particular, the lack of prior conceptualization necessary for the student to make sense of the ideas could explain the conceptual gap that exists in the process of building algebraic thinking. (Radford & Grenier, 1996). In our case, each relation linking idea and equal sign clearly underlies a conceptualization of mathematical objects, like equivalence and equality, and “the interaction between symbol and ideas should […] be seen as a network of relations forged by a human being in the intellectual pursuit both at the individual and societal level”. (Radford & Grenier, 1996, p. 254).

According to Carpenter, Falkner & Levi (1999), young children often understand much more than traditionally thought while adults find it difficult to conceptualize the appropriate algebra for the early childhood years. By viewing algebra as a strand in the curriculum from prekindergarten on, researchers and teachers would help define what algebra instruction can be for young children.

**ANALYSIS TOOLS**

**The Equal Sign and Algebraic Thinking**

According to Kieran (1996):

> Algebraic thinking can be interpreted as an approach to quantitative situations that emphasizes the general relational aspects with tools that are not necessarily letter-symbolic, but which can ultimately be used as cognitive support for introducing and for sustaining the more traditional discourse of school algebra. (Kieran, 1996, p. 275)

In this regard, several approaches can be used to develop early algebraic reasoning in students, the OME (2013) highlights two approaches for pre-school education:

a) Functional thinking which consists in analyzing regularities and patterns (numerical and geometrical) to identify a change and recognize the relationship between two sets of numbers (Beatty & Bruce, 2012).
b) Generalization of mathematics, which is based on the reasoning behind operations and properties associated with numbers (Carpenter, Franke, & Levi, 2003).

We place this article in the latter of these two approaches to go beyond the calculation of particular numbers and thus explore the mathematical structure underlying in particular the meaning of equality with respect to quantities (OME, 2013). Indeed, as pointed out by Kieran (2004), algebraic thinking requires a refocusing of the meaning of the equal sign. The process of interpretation of the symbol of equality is to associate it to human consciousness. The symbol of equality should evoke, among students, a particular thought to unveil a conceptual object, and point to specific features that belong to the significant field of meaning of such conceptual object (Ludlow & Walgamuth, 1998). From that point of view, the activities can be engaged without using the symbolic letter, and they can be specified at any time to include the symbolic letter to conceptualize a non-symbolic or pre-symbolic approach to algebraic thinking in elementary classes (Kieran, 2004). Our research is based on the development of ways of thinking without using any letter-symbolic algebra at all, where the objective is to study the symbol of equality as a relationship of equivalence and equality during early school years.

The Equal Sign in the Ontario Kindergarten Curriculum

According to Theis (2005), the OME (2008) recognizes two kinds of situations associated with the symbol of equality:

a) Equality situation: Relationship between two models of the same mathematical object.

b) Equivalence situation: Relationship resulting from ranking based on a single criterion: the quantity.

In addition, according to this document, three types of activities may be proposed to students in connection with situations of equality:

a) Example of a grouping scenario: Sonia has 3 red pencils and 5 blue pencils. How many colored pencils does Sonia have?

b) Example of a comparison scenario: John has two bags of marbles; one containing 2 green marbles, the other 3 yellow marbles. By comparing the quantity of his marbles to those of Thierry (5 marbles), John discovers that he has an equal number or 5.

c) Example of a transformation situation: a child plays with 5 marbles and wins 3, how many does he have now?

The OME distinguishes skills relating to the symbol of equality: acknowledge, clarify (explore, describe, express symbolically, offer hypotheses, generalize), create, re-establish (example: how many would be required to achieve the same quantities or a
balanced situation?) and maintain (example: if 2 towers are made up of 6 squares and 1 square is added to one of the towers, how would you maintain equality between the two towers?)

In this paper, we focus on the ability to “restore equality” for, as indicated by Taylor-Cox (2003) “By asking how many are needed to make the quantities the same or the situation fair, we incorporate the concept of equality, making algebraic thinking part of everyday life.” (Taylor-Cox, 2003, p.19)

**Conceptual Model of Equality and Equivalence**

In his thesis, Theis (2005) conducted a conceptual analysis of the relationship between equivalence and equality drawing on the model of understanding by Bergeron and Herscovics (1988). These authors underscore two logico-physical levels which imply an understanding of equivalence, where the child uses concrete objects for his reasoning, and logico-mathematical intelligence that requires numerical equality where the child reasons in terms of numbers. Furthermore, four types of understanding are differentiated for these two levels:

- **Intuitive understanding**: the child draws on visual perception and does not use mathematical procedures;
- **Procedural understanding**: the child uses early mathematical procedures;
- **Abstract understanding**: the child understands the construction of invariants, reversibility, the composition of transformation and generalization;
- **Formal understanding**: the child is able to use mathematical symbolism.

Based on this initial research model, we have combined with the levels and models of understanding, strategies, models and key concepts from the perspective of learning milestones by Dolk & Fosnot (2010). Count by touching an object with your finger only once or count starting from, are skills related to the process of developing knowledge of strategies or “process mapping” (Treffers, 1987). First introduced as reflexes, these initial learning schemes differentiate and coordinate themselves to create new strategies of exploration. According to Schifter and Fosnot “[K] key concepts are central ideas and frameworks for mathematics – principles define mathematical order”. (Dolk & Fosnot, 2010, p. 13), for example, equivalence, equality, quantity or number. Finally, according to Dolk & Fosnot:

“Models are representations of relationships that mathematicians have built over time by reflecting on how something can be transformed into another and generalizing ideas, strategies and representations from various contexts. […] When viewed from a certain angle, the models create conceptual maps used by
mathematicians to organize activities, solve problems or explore relationships” (Dolk & Fosnot, 2010, p. 84).
Consider drawings or connecting cubes as model examples.

To analyze skills “restore equality”, we sought to bring together different elements using the conceptual grid shown in Table 1:

<table>
<thead>
<tr>
<th>Levels</th>
<th>Types of understanding</th>
<th>Strategies</th>
<th>Key concepts</th>
<th>Models</th>
</tr>
</thead>
<tbody>
<tr>
<td>Logico-physical (equivalence on concrete objects)</td>
<td>Intuitive</td>
<td>Visual Perception</td>
<td>Quantity Equivalence</td>
<td>Ex. Connecting cubes To 10</td>
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<tr>
<td>Procedural</td>
<td></td>
<td>One to one Correspondence</td>
<td>Equivalence</td>
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</tr>
<tr>
<td>Abstract</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Logico-mathematical (numerical equality)</td>
<td>Procedural Pair # 1</td>
<td>Counting Double counting Pair # 1</td>
<td>Number Equality Pair # 1</td>
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<tr>
<td>Abstract</td>
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<td></td>
<td>Number Equality</td>
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<td>Number invariance</td>
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<td>Reversibility</td>
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<td>Operations Transformations</td>
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<td>Formal</td>
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<td>Number Equality</td>
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Table 1: Conceptual Grid for Skills Analysis “restore equality"

METHOD
Twenty-one 5-year old children participated in this study. This instrument has been used to study the findings of previous research (Theis, 2005). Types of understanding
among students about equality and equivalence have been defined and the instrument was developed based on this knowledge. The instrument consisted of two tasks that require students to “restore Equality”: A task was contextualized type problem situation (The history of Fafounet and Easter Egg Hunt, D’Aoust, 2011) and the other was decontextualized (with cubes, Squalli, 2007). This study was conducted in the regular school hours. Students had to complete tasks in a class period.

They completed answering the study questions in about 10 minutes. The researchers conducted research in person. The class teachers observe roles took and did not participate in the study.

**Task 1: The Story of Fafounet and the Easter Egg Hunt**

In this story, Fafounet the character, invites his neighbor Fafoundé to take part in a chocolate Easter egg hunt his mother had organized for him. The idea of the game is to respect his mother’s golden rule “you must find the Easter eggs together” and, at the end of the game, I will divide the eggs in equal parts between the two of you”. At the end of the story, Fafounet finds 8 chocolate eggs and Fafoundé 4.

The researcher reads the story to the students. Then, in teams of two where each student portrays a character, he hands out 8 cubes and 4 cubes to symbolize chocolate eggs found by the characters (Picture 1). Finally, he asks “what should you do to respect the golden rule?

![Picture 1: Cubes given to each student to represent the chocolate eggs in task 1.](image1)

**Task 2: Restoring Equality with Cubes**

The researcher gives this test to students individually. He shows them two groups of cubes, one composed of 10 cubes and the other of 8 cubes.
Then, he asks if there is an equal amount of cubes in each group, how many there are and, what should we do to have the same quantity in both clusters.

![Picture 2: Two groups of cubes given to students for task 2.](image)

RESULTS

Part 1: Contextualized Problem Situation

Understanding the Criteria for the Concepts of Equivalence and Equality

Like Theis (2005), we will undertake a conceptual analysis of the relationships of equivalence and equality based on the criteria established in his doctoral research. On that point, we will identify the types of understanding of equivalence and equality put forward by students to solve the problem situation proposed in our study.

First Pair of Students: procedural understanding (counting) at the logico-mathematical level

In the contextualized problem situation, two collections of cubes are presented to two little girls. The first collection contains 8 cubes while the second consists of 4. The challenge for the students is to share the cubes amongst themselves “equally” so that everyone has the “same amount.”

At first glance, the first little girl (the one who takes the lead in solving the problem) looks at the two sets and decides to give away two cubes to her teammate. Then, she counts her cubes (6) and her teammate’s (6) and determines that there is equality, that is to say each has the same amount of cubes.

When the researcher asks her how she achieved that outcome, the little girl says she immediately noticed that she had 8 and her teammate had 4. Accordingly, she knew she had to give some to her teammate. When the researcher asks whether she knew she had to give 2 she said she did not know the exact number but she knew she had to give some away.
To accomplish this, the fact that the little girl decides to give her teammate two cubes and uses counting to verify if the two girls have the same number of cubes is connected to a thought at the logico-mathematical level. According to Theis (2005), a thought at the logico-mathematical level is considered to be a reflection of an action performed by the subject and a reasoning in terms of the number, for example, when the subject must determine whether the same number characterizes various collections of objects. Furthermore, when the child is encouraged to verify a task or to verify the effectiveness of the operation he has just carried out, he performs an action called “a procedure.” In this regard, some procedures associated with the logico-mathematical level may be used by the child such as counting to determine if the same number is present in the groups proposed. In fact, in the problem situation, the little girl takes the time to count the cubes each one has after the action taken (the sharing of two cubes) to finally determine that there is the same number of items for herself and for her teammate. She counted (6) cubes for herself and (6) cubes for her teammate. She said “I have 6, and she has 6.” In her view, there was thus equality since the same number (6) was characteristic to both collections of objects and therefore both girls each had 6 cubes. The fact that the little girls could recognize the relationship of equivalence between the two sets of objects may be related to the development of algebraic thinking. “Functional thinking which consists in analyzing regularities and patterns (numerical and geometrical) to identify a change and recognize the relationship between two sets of numbers (Beatty and & Bruce, 2012).”

Second Pair of Students: intuitive understanding (visual perception) and procedural understanding (one to one correspondence; grouping of two cubes at a time) at the logico-physical level

A little girl and a little boy are the focal point in this activity. Initially, when she gets her cubes, the little girl counts them and says “I have four.” She also proposes to cut the cubes in pieces but changes her mind and instead suggests to “put an equal number.” The researcher asks them how they could achieve “an equal number.” The little boy offers to count “how many do we have.” The little girl disagrees saying “No, let’s put two and two.” To this end, she instantly separates cubes in pairs of two. The little boy agrees and starts dividing his cubes in groupings of two counting two by two (touching his cubes and those of his teammate) saying “I have two, you have 2, I have two more, you have two more, now two more for you (he gives her two cubes) and two more for me (he takes two more cubes)” and shouts: “it’s equal.” In that respect, it should be noted that no counting was performed to verify the outcome.

In the course of the activity, the little girl is initially interested in the number of cubes she has and for this reason, she counts them. Then, to meet the challenge laid down by the researcher, she proposes to classify the cubes by groups of two. In her mind, it is important to arrange their respective cubes in “an equal number” and the method
she has found is to put them in pairs of two. Her thought process points towards grouping the cubes “two by two”. She searches no further. Rather, it is the little boy who takes over and rises to the researcher’s challenge. To do this, he uses matching pairs to compare his cubes to those of his teammate. This procedure corresponds to a logico-physical thought. In fact, Theis (2005) points out: “At the logico-physical procedural understanding level, it is mathematical procedures such as one to one correspondence that enable the establishment of groupings.” (p.47). At the end of the activity, when the researcher asks how they managed to separate the “equal” the little girl explains that the little boy gave her two because he had “a lot” and she had “less.” She indicates that he “alone” had two. Her explanation of the situation shows that her understanding is more at the logico-physical level and that she is “intuitive” because she uses the terms “more and less” to compare the quantities in the two collections and not the numbers. Furthermore, she does not make a one to one correspondence, nor does she count to verify if the sharing is “equal.” She only has to look at the two collections to be satisfied with the outcome.

However, intuitively (for the little girl) and by making a one to one correspondence (2 X 2), therefore based on a logico-physical understanding, the children were able to establish a relationship of equivalence between two sets of objects and thus demonstrate an early development of algebraic thinking. To this end, Squalli (2007) states that it is “an algebra before the letter thereby putting the emphasis on thought and not mathematical content.” Here, children recognized that two quantities were equivalent without having to use numbers, and without calculating. All they did was base their reasoning on a logico-physical procedure like the one to one correspondence (2 X 2) which required the little boy to use a mathematical procedure, of giving one pair of two cubes to the little girl to achieve two equivalent quantities, more specifically, two collections of equivalent cubes. The recognition of the relationship of equivalence of two sets of objects, regardless of the chosen procedure, depends on the development of algebraic thinking. “Functional thinking which consists in analyzing regularities and patterns (numerical and geometrical) to identify a change and recognize the relationship between two sets of numbers (Beatty and & Bruce, 2012).”
Part 11: Decontextualized Problem Situation

This part addresses a decontextualized problem situation in which the researcher placed 10 cubes to the right of a board and 8 cubes to the left. First, the researcher asks each student individually if there are “as many” cubes on the right side as on the left side. Then, in a second phase, she asks them to ensure there are “as many” cubes in one set as the other. In the examples selected, several types of thoughts and procedures are used by children to judge a situation of equivalence and to restore the equivalence between two sets of cubes. Thus, of those children, some use intuitive thinking and rely on visual perceptions to judge equivalence while others use either counting, double counting or a measurement device made on site to establish equivalence between two collections of objects. In this regard, they have access to a thought that is more at the logico-mathematical level. However, among the examples presented, one child, a little girl appears to access more abstract thinking associated with an understanding of the conservation of equivalence. In fact, she knows that even though a transformation occurred in the number of objects in the sets, they are still equal since the same number of objects have been added on either side.

First Student: Entrenched intuitive thinking coupled with an auto-add strategy instead of sharing (adding and subtracting) to establish equivalence

In this situation, from the onset, the researcher asks the student if there are “as many” cubes on the right and left side of the board. The first student looks at both sets on the board and, pointing to the right, he says: “there.” When the researcher reviews the meaning of “as many” with him and asks if they are “equal” on both sides, the child looks at both sets again and says: “yes.” When the researcher asks him how he was able to know he says: “I thought in my head.”

In this situation, the child’s gestures underlie an understanding associated with intuitive thinking based on visual perceptions. It is at the heart of the logico-physical level and warrants no verification such as those associated with the one to one correspondence or counting found at the logico-mathematical level. The child therefore relies on what he sees. For this purpose, Theis (2005) points out that “Intuitive understanding which, by nature, appears only at the logico-physical level, is the first and also the most rudimentary form of understanding. Most often, it relies on visual perceptions and does not yet imply the use of procedures.” (p.49)

Moving on to the next phase of the activity, the researcher decides to go further and ask the child if he knows a way to check and make sure that it is “equal.” He says: “We could count.” To that end, the child counts 10 cubes on the right and 8 cubes on the left and, smiling, he says: “It’s not equal.” The researcher asks: “What can we do to make it “equal”? He answers: “We could add 1.” He then gets up to look for other cubes. He does not take those on the board. He takes three other cubes and places them on the board with the eight cubes on the left. When the researcher asks him if it is now “equal” he counts the cubes on both sides of the board. He counts (10) on the
right and (11) on the left, then decides to remove one cube. When the researcher asks if it is now “equal”, he replies “Yes.” At first glance, the child did not seem to need to use procedures to verify if the quantities of cubes were equivalent in both sets. Rather, it is the insistence of the researcher which led him to count the cubes. We can sense a benchmark here for a strategy to inform the child’s understanding with regards to the concept of equivalence and equality. Moreover, it is an “auto-add” strategy the child used as opposed to “remove and add” to achieve equivalence in both sets of objects. In fact, this auto-add strategy brought the child to do a single “calculation” (that of adding to the set with fewer objects) instead of two (remove from a set and add to the other). This process was, for the time being, probably more appropriate to the thought level of the child.

Second Student: Using a measurement model to verify equivalence

In this activity, the second student looks at both sets of cubes on the board and points to the right saying: “as many.” When the researcher reviews the meaning of the term “as many” with the child and asks her if both sides are “equal” and, looking at both sets, the child says: “No.” The researcher then asks: “What could we do in order to have as many on the right and on the left”? The child replies: “We could add some.” Then she begins to assemble all the cubes on the left side of the board and build a tower. She repeats the exercise with the cubes on the right and assembles them in a tower. She counts the cubes in the right tower and comes up with 8. At that moment she decides to remove 1 cube from the left tower and puts it back in a tray. She measures the two towers and realizes that the left tower is still higher. She seems surprised and to be certain, she turns the towers upside down and measures them again. Then she observes she must remove another cube from the left tower. Again she measures both towers smiling. The researcher asks her if they are now “equal” to which she nods “yes.” The researcher asks the child: “How did you know they were equal”? The child takes the two towers and measures them saying to the researcher: “Because I did that.” The researcher asks if she knows how many cubes are there in each tower. The child replies 9 adding that she did not count. At the end of the activity however, there were 8 cubes in each tower. This child’s thought seems to borrow from both the logico-physical and the logico-mathematical levels. In fact, the child feels the necessity to rely on her visual perceptions since she uses the measurement of the towers to verify equivalence. Nevertheless, she starts using procedures associated to the logico-mathematical level as they relate to counting. It appears, however, that there is a predominance of thought at the logico-physical level since the child seems satisfied when measuring the two towers and because she does not count again to ensure that both towers have an equal number of cubes.
Third Student: Understanding the Conservation of Equivalence

The third student looks at the two sets of cubes arranged on the board. Pointing to the left she says: “There are more cubes here” and pointing to the right she says: “Here, there are less.” The researcher asks: “What can we do have as many”? The child seems reflective and unresponsive. The researcher goes over the meaning of the term “as many.” Pointing to the left the child says: “You take away 4 here and then they are equal.” The child then proceeds to remove 4 cubes from the left. The researcher asks “Why 4”? The child replies: “It would be almost equal and we also remove 4 from there” pointing to the right. Then she changes her mind and says: “No, maybe 3, perhaps now it will be equal.” She said this looking once again at the number of cubes on the left and the number of cubes on the right. By examining both sides of the board, she decides to add one cube on the right saying: “Maybe one here and it would be equal.” The researcher asks her how she can be sure that both sides are “equal.” The child counts 6 cubes on the right and 6 cubes on the left. She says “they are equal.” Then the researcher asks the child what she could do with the cubes left in her hands. Note that the child kept 6 cubes in her hands. She divides them one by one in each of the sets. At this point the researcher asks if she is certain they are still “equal.” The child replies “yes.” The researcher asks how she knew, to which the child replies “because I added 1 cube to each set.” Then she counts 9 cubes in the set on the left and says, pointing to the right “I believe there are 9 here.” To this end, she counts 9 cubes. In this situation it appears that the child is already accessing an abstract thought connected to an understanding of the conservation of equivalence. Since she had actually counted 6 cubes to the right and 6 to the left and she had added cubes to each side, one by one, the child knew there was always a situation of equivalence. She knew that, in spite of the change in the number of objects in each set, they were always equal since the same number of objects had been added to both sides.

Fourth Student: Use of Two Types of Thinking: logico-mathematical thought related to counting followed by the logico-physical thought associated with one to one correspondence to verify equivalence

The researcher asks the child if there are “as many” cubes here (pointing to the left) as here (pointing to the right). The child counts the cubes in both sets in her head. She says 9 on the left side and 7 on the right side. The researcher asks if there are “as many” on this board and that board. The child points to the left. When the researcher verifies the meaning of the term “as many” and asks if there are “as many” she replies “no.” The researcher asks what she could do to make them “equal.” The child counts both sets again (several times) and says 10 and 8. Then she makes a one to one correspondence saying “1-1, 2-2, 3-3… up to 8 and puts one on the right as she says: “8 and 8.” She counts again (several times because of miscounting) then reaches 9
and 9 on both sides. The researcher asks if both sides are “equal” now. She replies “yes.” We note here that the child used several procedures at her disposal to confirm a situation of equivalence. We assume that the fact that she often miscounted may have prompted her to use a procedure whereby she could refer to something more tangible, that is, visual perceptions or one-to-one correspondence.

CONCLUSION

From a child’s first years at school the OME suggests that the development of algebraic thinking implies the understanding of equality and equivalence. Children should study these situations using different models, concepts and strategies. Most of the time, however, equality and equivalence are taught from a numerical perspective and the symbol of equality is introduced early on. We have therefore proposed tasks which focus on the meaning of the symbol of equality prior to its formalization.

On the one hand, the recitation of numbers demonstrates that, in all the situations presented, it seems that many children were able to establish equivalence between two quantities (sets) of objects using different procedures to achieve results. We believe, however, that any child among those studied clearly acquired an early algebraic reasoning via the process of generalization. What comes to mind here is the little girl in situation #3 who made the connection between the changes found in the sets and the conservation of equivalence in each set. Presumably, her thought presupposes an ability to build relationships and to generalize. In that sense, and in that situation, the little girl knows that regardless of the number of cubes she adds, if that number is the same in both sets of objects, the quantities will always remain equivalent.

On the other, the recitation of numbers is very powerful for the child throughout the phase of experimentation and even the child who has difficulty touching each object without skipping or double touching, will have no difficulty respecting the established order in the numbers of the counting rhyme. As Herscovics, Bergeron and Bergeron (1987a) will tell you: “Children may often know how to recite the sequence without necessarily coordinating it correctly in their enumeration process.” (p.348). Reciting numbers has been a factor underpinning counting throughout the activities proposed as a part of our research project. It would be easy for us to deduce that the recitation of numbers is an oral model the child uses to support the strategy of synchrony: one word, one object, essential to the procedural logico-mathematical understanding in children.

The reference model by Dolk & Fosnot (2010) displays a range of strategies, models and big ideas which we could link to intuitive understanding, procedural or abstract logico-physical and logico-mathematical of the hybrid model used by Theis (2005) and developed by Herscovics & Bergeron (1982). Finally, the passage on intuitive understanding (related to perception) appears to be facilitated by procedural understanding (related to strategies) and the latter could be used to access abstract
understanding (the development of concepts). These types of understanding seem necessary to the conceptualization of situations of equality and equivalence and indeed give meaning to the symbol of equality.

REFERENCES


